

Galactic dynamics

Chandrasekhar dynamical friction formula

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If we analyze the movements of the stars of a galactic population, we see that they are collisionless: both geometrical and strong encounters are negligible, with a relaxation time greater than Hubble time, while weak encounters could be important in globular cluster and open cluster systems. The more stars we consider, the longer it takes to have a weak encounter, a time that we call *relaxation time*. With this relation it's easy to see that with the global clusters, and especially with the open clusters, this ratio becomes critical, so: while galaxies are *collisionless systems*, in clusters systems the encounters could play a key role.

By studying the general case of galactic encounters, it's possible to apply the *Chandrasekhar dynamical friction formula*¹ (originally derived for a uniform background distribution) to a body of mass M_{sat} orbiting in a spherical system with density $\rho(r)$ and total mass $M_{\text{host}} \gg M_{\text{sat}}$. Let us assume that the mass density distribution of the host system is that of a *singular isothermal sphere*, suitable for studying the dynamics of stellar flows, and obtain this mass density equation starting from a distribution function $f(\xi)$ (see below) like the following:

$$f(\xi) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{\xi}{\sigma^2}} ,$$

where σ is a parameter which we will identify as dispersion velocity, ρ_1 is a scale density parameter, $\xi \equiv \psi - \frac{1}{2}v^2$ is defined as relative energy with ψ as potential (in the energy formulas we consider unity masses).

The mass density is proportional to the volumetric number density of stars $\nu = \int f d^3\vec{v}$, where $f(\vec{x}, \vec{v}, t)$ is the distribution function (or phase space density) defined in such way that $f(\vec{x}, \vec{v}, t) d^3\vec{x} d^3\vec{v}$ gives us the number of stars within $d^3\vec{x}$ with velocity within $d^3\vec{v}$. Therefore we obtain:

$$\rho = \int_0^\infty f d^3\vec{v} = 4\pi \int_0^\infty f(\xi) v^2 dv = \frac{4\pi\rho_1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{\psi}{\sigma^2}} \int_0^\infty e^{-\frac{v^2}{2\sigma^2}} v^2 dv .$$

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¹A note of clarification from the work [Chandrasekhar, 1943]: «To avoid misunderstandings we shall make some remarks (which are otherwise obvious) concerning the reason for introducing the new notion of dynamical friction avoiding the usage of the term "viscosity". First, the physical ideas underlying the concepts of dynamical friction and viscosity are quite distinct: thus, while the "coefficient of dynamical friction" refers to the systematic deceleration which individual stars experience during their motion, "viscosity", as commonly understood, refers to the sheering force exerted by one element of gas on another. Second, dynamical friction is an exact notion expressing the systematic decelerating effect of the fluctuating field of force acting on a star in motion, in contrast to viscosity, which, as a concept, is valid only when averaged over times which are long compared to the time of relaxation of the system and over spatial dimensions which are large compared to the mean free paths of the individual molecules. Thus, while the introduction of dynamical friction in stellar dynamics presents no difficulty, the circumstances are very different for a rational introduction of "viscosity" in the subject.»

Substituting $t^2 = v^2/2\sigma^2$ and $dv = \sqrt{2}\sigma dt$ in the last term, we get:

$$\rho = \frac{4\rho_1}{\sqrt{\pi}} e^{\frac{\psi}{\sigma^2}} \int_0^\infty e^{-t^2} t^2 dt = \frac{4\rho_1}{\sqrt{\pi}} e^{\frac{\psi}{\sigma^2}} \frac{1}{2} \frac{\sqrt{\pi}}{2} \quad \text{that is} \quad \rho = \rho_1 e^{\frac{\psi}{\sigma^2}}.$$

If we rewrite it as $\psi = \sigma^2 \ln\left(\frac{\rho}{\rho_1}\right) = \sigma^2 \ln(\rho) - \sigma^2 \ln(\rho_1)$, differentiating we get $d\psi = \sigma^2 d\ln(\rho)$. Then we can rewrite the Poisson equation $\nabla^2\Phi = 4\pi G\rho$ for the potential as:

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) &= -4\pi G\rho \quad \text{that is} \\ \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\ln(\rho)}{dr} \right) &= -\frac{4\pi G}{\sigma^2} \rho. \end{aligned} \quad [1]$$

We are looking for a solution in the form $\rho = cr^{-b}$, therefore we rewrite it as $\ln(\rho) = \ln(c) - b\ln(r)$, and differentiate it as $d\ln(\rho) = -b d\ln(r)$, then we apply these values to the Poisson equation [1], obtaining the two parameters of the exponential as $b = 2$, and $c = \frac{\sigma^2 b}{2\pi G}$, and finally the mass density equation:

$$\rho(r) = \frac{\sigma^2}{2\pi G} \frac{1}{r^2} = \frac{v_c^2}{4\pi G} \frac{1}{r^2}, \quad [2]$$

with the parameter $\sigma = v_c/\sqrt{2}$ substituting the value ρ from the first term of [2], where v_c is the (radius independent) circular velocity:

$$v_c^2(r) = \frac{G}{r} M(r) = \frac{G}{r} \int_0^r 4\pi r'^2 \rho dr' = \frac{G}{r} \int_0^r 4\pi r'^2 \frac{\sigma^2}{2\pi G r'^2} dr' = \frac{G}{r} \frac{2\sigma^2}{G} r.$$

In this case, for a satellite of mass M_{sat} orbiting on a circular orbit (with $v = v_c$) at radius r , following [Cimatti et al., 2020], the Chandrasekhar dynamical friction formula reads:

$$\frac{dv}{dt} = -\text{const} \ln(\Lambda) \frac{GM_{\text{sat}}}{r^2} = -2\pi \text{const} \ln(\Lambda) \frac{G^2 \rho M_{\text{sat}}}{\sigma^2},$$

where the *constant* comes from the statistical *error function*.

Therefore the dynamical friction timescale:

$$t_{\text{fric}} \equiv \frac{v}{|dv/dt|} = \frac{1}{\sqrt{2}\pi \text{const} \ln(\Lambda)} \frac{\sigma^3}{G^2 \rho M_{\text{sat}}}, \quad [3]$$

is inversely proportional to M_{sat} and ρ , for given velocity dispersion of the host. Applying [2] to [3] we obtain:

$$t_{\text{fric}} = \frac{\sqrt{2}}{\text{const} \ln(\Lambda)} \frac{\sigma r^2}{GM_{\text{sat}}}.$$

This means that in a time span of a few Gyr, for instance with the values of our assumptions applied to the Milky Way, the orbit of a massive satellite has substantially reduced, while the orbit of lower mass satellites (such as globular clusters) has not been significantly modified by dynamic friction.

References

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