

II Jeans equation in cylindrical coordinates

We will derive the II Jeans equation (the Jeans equations are a set of partial differential equations that describe the motion of a collection of stars in a gravitational field) from the collisionless Boltzman equation (CBE) in cylindrical coordinates, therefore let's get that one first.

CBE in scalar and in vector notation reads:

$$\begin{aligned} \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} &= 0 \\ \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla} \Phi \cdot \frac{\partial f}{\partial \vec{v}} &= 0 \end{aligned} \quad [1]$$

it comes from lagrangian derivative, derivable itself from the Liouville theorem:

$$\frac{d f}{d t} = \frac{\partial f}{\partial t} + \sum_{i=1}^6 \dot{\vec{W}}_i \frac{\partial f}{\partial W_i} = 0$$

where $\vec{W} = (\vec{x}_i, \vec{v}_i)$ is the position and

$$\dot{\vec{W}} = (\dot{\vec{x}}_i, \dot{\vec{v}}_i) = (\vec{v}, -\vec{\nabla} \Phi) \quad [2]$$

is the velocity in the phase space, where x_i and v_i are independent coordinates, and where the acceleration of $\vec{v}$ is represented by the derivative of the smooth potential Φ that we consider in our sample galaxy $\left[\vec{\nabla} \Phi = \frac{\partial \Phi}{\partial \vec{x}} \right]$.

If we consider the cylindrical coordinates $(R, \varphi, z), (\hat{R}, \hat{\varphi}, \hat{z})$, the velocity reads:

$$\vec{v} = (\dot{R} \hat{R} + R \dot{\varphi} \hat{\varphi} + \dot{z} \hat{z}) = (v_R \hat{R} + v_\varphi \hat{\varphi} + v_z \hat{z}) \quad [3]$$

where $\dot{R} = v_R$ $\dot{\varphi} = \frac{v_\varphi}{R}$ $\dot{z} = v_z$.

If we write the CBE in explicit form, from [1] we get:

$$\frac{\partial f}{\partial t} + \dot{R} \frac{\partial f}{\partial R} + \frac{\dot{\varphi}}{R} \frac{\partial f}{\partial \varphi} + \dot{z} \frac{\partial f}{\partial z} - \dot{v}_R \frac{\partial f}{\partial v_R} - \dot{v}_\varphi \frac{\partial f}{\partial v_\varphi} - \dot{v}_z \frac{\partial f}{\partial v_z} = 0 \quad [4]$$

In cylindrical coordinates, from [3] the acceleration reads:

$$\vec{a} = \frac{d \vec{v}}{d t} = (\ddot{R} - R \dot{\varphi}^2) \hat{R} + (2 \dot{R} \dot{\varphi} + R \ddot{\varphi}) \hat{\varphi} + \ddot{z} \hat{z}.$$

Considering the smooth potential $\Phi(R, \varphi, z)$, its components are parallel to the unity vectors $(\hat{R}, \hat{\varphi}, \hat{z})$, therefore:

$$-\vec{\nabla} \Phi = \frac{\partial \Phi}{\partial R} \hat{R} + \frac{1}{R} \frac{\partial \Phi}{\partial \varphi} \hat{\varphi} + \frac{\partial \Phi}{\partial z} \hat{z} = \vec{a}.$$

Thus we can write the acceleration as follows (ref. [2]):

$$\begin{aligned}\dot{v}_R - R \dot{\varphi}^2 &= -\frac{\partial \Phi}{\partial R} & \dot{v}_R &= \frac{v_\varphi^2}{R} - \frac{\partial \Phi}{\partial R} \\ 2 v_R \dot{\varphi} + R \dot{v}_\varphi &= -\frac{1}{R} \frac{\partial \Phi}{\partial \varphi} & \dot{v}_\varphi &= -\frac{1}{R} \left(\frac{v_R v_\varphi}{R} + \frac{1}{R} \frac{\partial \Phi}{\partial \varphi} \right) \\ \dot{v}_z &= -\frac{\partial \Phi}{\partial z} & \dot{v}_z &= -\frac{\partial \Phi}{\partial z}\end{aligned}$$

Therefore, remembering the expression [3] for the velocity, the CBE in cylindrical coordinates [4] reads:

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\varphi}{R} \frac{\partial f}{\partial \varphi} + v_z \frac{\partial f}{\partial z} - \left(\frac{v_\varphi^2}{R} - \frac{\partial \Phi}{\partial R} \right) \frac{\partial f}{\partial v_R} + \cancel{R} \frac{1}{\cancel{R}} \left(\frac{v_R v_\varphi}{R} + \frac{1}{R} \frac{\partial \Phi}{\partial \varphi} \right) \frac{\partial f}{\partial v_\varphi} + \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0 .$$

Now we can obtain the Jeans equation from CBE multiplying the above equation by v_R and integrating over all velocities, where $v = \int f d^3 \vec{v}$:

$$\begin{aligned}& \frac{\partial}{\partial t} \int f v_R d^3 \vec{v} + \frac{\partial}{\partial R} \int f v_R^2 d^3 \vec{v} + \frac{\partial}{\partial \varphi} \frac{1}{R} \int f v_R v_\varphi d^3 \vec{v} + \frac{\partial}{\partial z} \int f v_R v_z d^3 \vec{v} + \\ & - \frac{\partial}{\partial v_R} \int f \frac{v_R v_\varphi^2}{R} d^3 \vec{v} + \frac{\partial}{\partial v_R} \int f v_R \frac{\partial \Phi}{\partial R} d^3 \vec{v} + \frac{\partial}{\partial v_\varphi} \int f \frac{v_R^2 v_\varphi}{R} d^3 \vec{v} + \\ & + \frac{\partial}{\partial v_\varphi} \frac{1}{R} \int f v_R \frac{\partial \Phi}{\partial \varphi} d^3 \vec{v} + \frac{\partial}{\partial v_z} \int f v_R \frac{\partial \Phi}{\partial z} d^3 \vec{v} = 0\end{aligned}$$

doing the integrals and defining $\bar{v} = \int f v d^3 \vec{v}$ and $\overline{v_R v_z} = \frac{1}{v} \int f v_R v_z d^3 \vec{v}$ we can write:

$$\begin{aligned}& \frac{\partial}{\partial t} (v \bar{v}_R) + \frac{\partial}{\partial R} (v \bar{v}_R^2) + \cancel{\frac{\partial}{\partial \varphi} \frac{(v \bar{v}_R \bar{v}_\varphi)}{R}} + \frac{\partial}{\partial z} (v \overline{v_R v_z}) - \frac{\partial}{\partial v_R} \frac{(v \bar{v}_R \bar{v}_\varphi^2)}{R} + \\ & + \frac{\partial}{\partial v_R} (v \bar{v}_R) \frac{\partial \Phi}{\partial R} + \frac{\partial}{\partial v_\varphi} \frac{(v \bar{v}_R^2 \bar{v}_\varphi)}{R} + \cancel{\frac{\partial}{\partial v_\varphi} \frac{1}{R} (v \bar{v}_R \frac{\partial \Phi}{\partial \varphi})} + \cancel{\frac{\partial}{\partial v_z} (v \bar{v}_R \frac{\partial \Phi}{\partial z})} = 0 ,\end{aligned}$$

and doing the derivatives, dropping some terms with the derivative towards different independent variables and the derivative respect φ for symmetry, we can finally write:

$$\frac{\partial}{\partial t} (v \bar{v}_R) + \frac{\partial}{\partial R} (v \bar{v}_R^2) + \frac{\partial}{\partial z} (v \overline{v_R v_z}) - \frac{v \bar{v}_\varphi^2}{R} + v \frac{\partial \Phi}{\partial R} + v \frac{v \bar{v}_R^2}{R} = 0$$

or, in the usual form for the II Jeans equation in cylindrical coordinates:

$$\frac{\partial}{\partial t} (v \bar{v}_R) + \frac{\partial}{\partial R} (v \bar{v}_R^2) + \frac{\partial}{\partial z} (v \overline{v_R v_z}) + v \left(\frac{\bar{v}_R^2 - \bar{v}_\varphi^2}{R} + \frac{\partial \Phi}{\partial R} \right) = 0 .$$